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THE FUNDAMENTAL SUPERSYMMETRY CHALLENGE REMAINS[1]

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ABSTRACT

In the following, we will review the fundamental problem that prevents a complete understanding of a theory of supersymmetrical field representations and describe its possible relation to a similar problem facing superstring/M-theory.

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1 INTRODUCTION

There are some problems in mathematics that have taken centuries to solve. Perhaps the best known recent example of this was the proof of Fermat's Last Theorem. Through the extraordinary insight and persistence of Andrew Wiles we now possess a proof to what had been a 350-year puzzle.

Is it time to wonder whether theoretical physics is capable of generating such problems?

In in area of supersymmetry, for 1/14 as long as the Fermat Puzzle lasted, there has been such a problem. Many years ago one of the present authors (SJG) became fascinated with the question of "Why is it that in most theories involving supersymmetry, we not able to describe them in a way that is independent of their dynamics?" This is an alternate statement of the notorious "auxiliary field problem."

This question still does not possess any known answer. While this question has been largely overlooked, we remain convinced that it is a key one for any theory which claims to provide a fundamental description of our universe. The latest realm where such a deadlock remains is superstring/M-theory.

Any truly covariant formulation of superstring/M-theory ought to permit us to understand its symmetries as readily as does general relativity. Thus our less than complete understanding of the representation theory of supersymmetry, in our opinion, is why our most cherished dream of a covariant formulation of a "final" theory remains out of our collective grasps.

2 THE OFF-SHELL SUSY PROBLEM

The statement of the problem is rather simple. The fact that it has remained without a general answer for over thirty years suggests that the answer is not. Consider a set of fields $\{\varphi_i\}$ where the index i $(= 1,...,s_{max})$ counts the number of fields and they may be arbitrary representations of a D-dimensional Minkowski, Euclidean or any other signature metric associated with a D-dimensional space. We introduce a variation operation denoted by $\delta_Q(\epsilon)$ that depends on a Grassmann parameter ϵ. The Grassmann parameter ϵ^α should transform as the spinorial representation of the D-dimensional space.

We say that a set of fields forms a *off-shell* representation of supersymmetry when

$$[\,\delta_Q(\epsilon_1)\ ,\ \delta_Q(\epsilon_2)\,]\,\varphi_i = \ i\ < \epsilon_1 \gamma^{\underline{a}} \epsilon_2 > \ \partial_{\underline{a}}\,\varphi_i \quad , \tag{1}$$

where we assume some appropriate inner product exists in the space of spinors in order to make a contraction meaningful on the RHS above. As simple as this statement may be, it is satisfied in a very small number of the known constructions involving supersymmetry.

Note that the definition above is independent of the issue of an action. As a second step, we may consider that set of fields φ_i appear in a Lagrangian $\mathcal{L}(\varphi)$ such that under the action of the variation $\delta_Q(\epsilon)$, the Lagrangian is changed by a total derivative.

It is more often the case that one starts with a set of fields $\{\widehat{\varphi}_i\}$ (where the index $i = 1,...,s_{min}$) that satisfy,

$$[\,\delta_Q(\epsilon_1)\ ,\ \delta_Q(\epsilon_2)\,]\,\widehat{\varphi}_i =\ i\ < \epsilon_1 \gamma^{\underline{a}} \epsilon_2 > \partial_{\underline{a}}\,\widehat{\varphi}_i\ +\ \epsilon_1^{\alpha}\,\epsilon_2^{\beta} \mathcal{F}_{\alpha\beta\, i}\,(\widehat{\varphi}) \quad , \tag{2}$$

for some set of functions $\mathcal{F}_{\alpha\beta\, i}$. Typically, these functions are such that they also arise uniformly from the variation of some Lagrangian $\mathcal{L}(\widehat{\varphi})$. In this case the representation $\widehat{\varphi}_i$ is said to be an "on-shell" representation of supersymmetry.

Not all of the fields in the set $\{\varphi_i\}$ propagate Cauchy data. The fields which propagate Cauchy data are called "the propagating fields." In fact only a subset denoted by $\{\overline{\varphi}_i\}$ will do so. If the set $\{\overline{\varphi}_i\}$ is isomorphic to the set $\{\widehat{\varphi}_i\}$, then we say that the $\{\varphi_i\}$ set is "an off-shell extension" of the latter. The set of fields in $\{\varphi_i\}$ with indices $i = (s_{min} + 1), ...s_{max}$ are called "auxiliary fields". The manner in which these appear in the Lagrangian implies that their equations of motion are *solely* algebraic.

The off-shell supersymmetry problem actually consists of *two distinct* but inter-related problems. The first problem may be stated as Problem (A.);

"Without regard to the existence of an action *and* with the smallest number and spin of auxiliary fields, for a given set of propagating fields find a set of propagating and auxiliary fields for which (1.) is satisfied."

This problem is one of representation theory. It has no relation whatsoever to dynamics and can be studied accordingly. There is a second problem. Problem (B.);

"Given the existence of a set of fields satisfying (1.) are these sufficient to permit the existence of a Lagrangian?"

In most interesting supersymmetrical theories these questions become obscured by the existence of additional symmetries (local and global) . In the presence of these, equation (1) becomes modified to read,

$$[\,\delta_Q(\epsilon_1)\;,\;\delta_Q(\epsilon_2)\,]\,\varphi_i = \;i\; < \epsilon_1 \gamma^{\underline{a}} \epsilon_2 > \;\partial_{\underline{a}}\,\varphi_i \;+\; \delta_{symmetry}\,\varphi_i \quad , \tag{3}$$

which complicates the analysis.

These questions are not regarded as being of great importance throughout most of the literature. Progresses in supersymmetry, supergravity, superstrings and M-theory have continued without a basic resolution of these problems...apparently.

3 OFF-SHELL SPINNING PARTICLES

Some years ago, [1,2] we began an avenue of attack on the off-shell supersymmetry problem by asking whether it was possible to find a large *class* of theories where the off-shell supersymmetry problem might be resolved? In this way, we were driven to study, perhaps the simplest of supersymmetrical systems, the off-shell spinning particles (we refer the reader to [1,2] for a more complete list of references). Our efforts were rewarded. Using certain Clifford algebra representations, we were able to show that there exist a solution to Problem (A.) and perhaps also to Problem (B.).

To understand the nature of the our proposed solution, it is first necessary to introduce a class of real Clifford algebras. We may denote these real N linearly independent $d \times d$ matrices by γ^I with $I = 1, \ldots, N$ which satisfy

$$\gamma^{\rm I}\,\gamma^{\rm J} \;+\; \gamma^{\rm J}\,\gamma^{\rm I} \;=\; -\;2\delta^{\rm IJ}\,\mathbf{I} \quad . \tag{4}$$

However, we are not just interested in all finding representations that satisfy this condition. We wish to restrict ourselves to the subset of these algebras that also admit the existence of another matrix denoted by Q that satisfies the relations

$$Q^2 \;=\; 1 \quad , \quad \gamma^{\rm I}\,Q \;+\; Q\,\gamma^{\rm I} \;=\; 0 \quad . \tag{5}$$

We call this subset of the real Clifford algebras the "$\mathcal{GR}(N, d)$ algebras." For a fixed value of N, there exists a smallest value of d (denoted by d_N) such that one can construct the $N + 1$ linearly matrices γ^I and Q. The relation between d_N and N is simply expressed in terms of the Radon-Hurwitz function [3]

$$d_N \;=\; 2^{4m+1}\,F_{\mathcal{RH}}(N) \quad , \tag{6}$$

that was written in tabular form in [1,2] (see these works for notational details). Due the result in (5) it follows that projection operators can be constructed

$$P_{\pm} ~=~ \tfrac{1}{2}(I ~\pm~ Q) \quad , \tag{7}$$

from which it further follows that

$$P_{+}\,\gamma^{\rm I}\,P_{+} ~=~ P_{-}\,\gamma^{\rm I}\,P_{-} ~=~ 0 \quad . \tag{8}$$

The remaining parts of the $\gamma^{\rm I}$ matrices may be denoted by the symbols

$$L_{I} ~=~ P_{+}\gamma^{\rm I}P_{-} \quad , \quad R_{I} ~=~ P_{-}\gamma^{\rm J}P_{+} \quad . \tag{9}$$

In particular, there are two types of "spinor" indices associated with the quantities L^{I} and R^{I},

$$L^{\rm I} ~\equiv~ (L^{\rm I})_{i\,\hat{k}} \quad , \quad R^{\rm I} ~\equiv~ (R^{\rm I})_{\hat{k}\,i} \quad , \tag{10}$$

with each type of index taking on values from 1 to d.

The object Q effectively plays the role of a chirality matrix. Accordingly, the analog of the usual "dotted-undotted" notation of the usual Van der Waerden formalism may be applied to these systems.

$$\begin{aligned} (L^{(\rm I}\,R^{\rm J)})_{i\,k} ~&=~ -\,2\delta^{\rm IJ}\,\delta_{i\,k} \quad , \\ (R^{(\rm I}\,L^{\rm J)})_{\hat{i}\,\hat{k}} ~&=~ -\,2\delta^{\rm IJ}\,\delta_{\hat{i}\,\hat{k}} \quad . \end{aligned} \tag{11}$$

The product space of all possible $\wedge$-products of the γ^{I}-matrices is decomposed under the action of the projection operators $P_{\pm}$ into four sub-spaces,

$$\begin{aligned} \{\mathcal{U}\} ~&=~ \{P_{+},\, P_{+}\gamma^{\rm IJ}P_{+} \,, \dots,\, P_{+}\gamma^{[N]}P_{+}\} \\ \{\mathcal{M}\} ~&=~ \{P_{+}\gamma^{\rm I}P_{-} \,, \dots,\, P_{+}\gamma^{[N-1]}P_{-}\} \\ \{\widehat{\mathcal{U}}\} ~&=~ \{P_{-},\, P_{-}\gamma^{\rm IJ}P_{-} \,, \dots,\, P_{-}\gamma^{[N]}P_{-}\} \\ \{\widehat{\mathcal{M}}\} ~&=~ \{P_{-}\gamma^{\rm I}P_{+} \,, \dots,\, P_{-}\gamma^{[N-1]}P_{+}\} \end{aligned} \tag{12}$$

The relevance of this discussion to the problem of the off-shell representation of spinning particles has been suggested by the following observation in the works of [1,2]. (There are some subtleties in the structures of $\mathcal{U}$, $\mathcal{M}$, $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{M}}$ for arbitrary N. These will be discussed more completely elsewhere.)

Associate with each of the Clifford algebra elements of $\mathcal{U}$ and $\widehat{\mathcal{M}}$ a set of 1D fields.

$$\begin{aligned} {\rm F} : \{\mathcal{U}\} ~&\to~ \{X(\tau),\, F^{\rm IJ}(\tau)\,, \dots,\, F^{[N]}(\tau) ~\} \\ {\rm F} : \{\widehat{\mathcal{M}}\} ~&\to~ \{\Psi^{\rm I}(\tau),\, \Lambda^{{\rm I}_i\,{\rm I}_2\,{\rm I}_3}(\tau)\,, \dots,\, \Lambda^{[N-1]}(\tau) ~\} \quad . \end{aligned} \tag{13}$$

In particular, the quantities $X(\tau)$ and $\Psi^I(\tau)$ may be identified with the position vector and NSR fermions of a 1D-spinning particle model. The remaining fields are auxiliary fields. We call the multiplet of fields in this construction, "the universal spinning particle multiplet" or USPM. We will also later introduce its canonically conjugate momentum multiplet.

Our first proposition is that the USPM and its canonically conjugate momentum multiplet provide a representation of the algebra in (1). To prove this, we write a set of supersymmetry variations

$$\begin{aligned} \delta_Q \, \mathrm{X} &= i\epsilon^{\mathrm{I}} \, \Psi_{\mathrm{I}} \quad , \\ \delta_Q \, \Psi_{\mathrm{I}} &= -2 \, [\, \epsilon_{\mathrm{I}} \, (\partial_\tau \mathrm{X}) \; + \; d^{-1} \epsilon^{\mathrm{J}} (f_{\mathrm{IJ}})_i{}^j \mathcal{F}_j{}^i \,] \quad , \\ \delta_Q \, \mathcal{F}_i{}^j &= i\epsilon^{\mathrm{I}} \, (f_{\mathrm{IK}})_i{}^j (\partial_\tau \Psi_{\mathrm{K}}) \; + \; i\epsilon^{\mathrm{K}} \, (L_{\mathrm{K}})_i{}^{\hat{k}} \Lambda_{\hat{k}}{}^j \quad , \\ \delta_Q \, \Lambda_{\hat{k}}{}^j &= 2\epsilon^{\mathrm{K}} \, \partial_\tau \, [\, (R_{\mathrm{K}})_{\hat{k}}{}^l \mathcal{F}_l{}^j \; + \; d^{-1} (R^{\mathrm{I}})_{\hat{k}}{}^j (f_{\mathrm{IK}})_j{}^l \mathcal{F}_l{}^j \,] \quad , \end{aligned} \tag{14}$$

and where $f_{\mathrm{IJ}} \equiv P_+ \gamma_{\mathrm{IJ}} P_+$, $\mathcal{F}_i{}^i = (L_{\mathrm{I}})_j{}^{\hat{k}} \Lambda_{\hat{k}}{}^j = 0$. Also in writing these transformation laws, we have introduced Duffin-Kummer-Petiau fields denoted by $\mathcal{F}_i{}^k$ and $\Lambda_{\hat{k}}{}^j$. These collectively include all the auxiliary fields. It now becomes an exercise to show that these supersymmetry variation satisfy (1) while placing no restrictions on any of the fields of the type present in (2). We thus regard this as part of a proof that there is a solution to Problem (A.) for the spinning particle models when we pick $d = d_N$.

However, the solution described above does not necessarily solve problem (B.). In fact, our studies indicate that given the USPM *alone* and in the *general case*, it is *not* possible to write an action that leads to the appropriate equations of motion except for the cases of $N = 1$,2 and 4. In particular, it can be shown that the representation described by the USPM in the special cases of $N = 1$,2 and 4 provides a *reducible* supersymmetry presentation.

For the values $N = 1$,2 and 4, there is a truncation that may be performed to obtain a smaller representation. The reason for the exceptional nature of these cases can be traced back to the supersymmetry variation of the fermionic DKP field in the USPM in the special case of $N = 4$ (the other two are its truncations). If we set this field to zero, then consistency of the supersymmetry implies

$$0 \; = \; 2\epsilon^{\mathrm{K}} \, \partial_\tau \, [\, (R_{\mathrm{K}})_{\hat{k}}{}^l \mathcal{F}_l{}^j \; + \; d^{-1} (R^{\mathrm{I}})_{\hat{k}}{}^k (f_{\mathrm{IK}})_k{}^l \mathcal{F}_l{}^j \,] \quad . \tag{15}$$

It is a remarkable fact that there are non-vanishing solutions for $\mathcal{F}_i{}^j$ in the cases only for $N = 1$,2 and 4. For the exceptional $N = 4$ case, the action for the spinning particle is of the form

$$\mathcal{S}_{ex}^{N=4} = \int d\tau \, \Big[\, \tfrac{1}{2} (\, \partial_\tau \mathrm{X}) (\, \partial_\tau \mathrm{X}\,) \; + \; i \tfrac{1}{2} \Psi_{\mathrm{I}} \partial_\tau \Psi_{\mathrm{I}} \; + \; \tfrac{1}{4} F_{\mathrm{IJ}} \, F_{\mathrm{IJ}} \, \Big] \quad , \tag{16}$$

where the auxiliary field satisfies $F_{\rm IJ} = \frac{1}{2}\xi\epsilon_{\rm IJKL}F_{\rm KL}$ for $\xi = \pm 1$. The component field $F_{\rm IJ}$ is the part of $\mathcal{F}_i{}^j$ that lies in the null-space defined by (15). The $N = 4$ action is invariant under the supersymmetry variations

$$
\begin{aligned}
\delta_Q \mathrm{X} &= i\,\epsilon_{\rm I}\,\Psi_{\rm I} \quad , \\
\delta_Q \Psi_{\rm I} &= -\,\epsilon_{\rm I}\,(\,\partial_\tau \mathrm{X}\,) \;+\; \epsilon_{\rm K} F_{\rm KI} \quad , \\
\delta_Q F_{\rm IJ} &= -\,i\,\tfrac{1}{2}\,[\,\epsilon_{\rm I}\partial_\tau\Psi_{\rm J} \;-\; \epsilon_{\rm J}\partial_\tau\Psi_{\rm I} \;+\; \xi\epsilon_{\rm K}\epsilon_{\rm IJKL}(\,\partial_\tau\Psi_{\rm L}\,)\,] \quad .
\end{aligned} \tag{17}
$$

The $N = 2$ exceptional truncation of this is given by

$$
\mathcal{S}_{ex}^{N=2} = \int d\tau \,\Big[\, \tfrac{1}{2}(\,\partial_\tau \mathrm{X}\,)(\,\partial_\tau \mathrm{X}\,) \;+\; i\tfrac{1}{2}\Psi_{\rm I}\partial_\tau\Psi_{\rm I} \;+\; \tfrac{1}{2}F\,F \,\Big] \quad , \tag{18}
$$

with transformation laws given by

$$
\begin{aligned}
&\delta_Q \mathrm{X} = i\,\epsilon_{\rm I}\,\Psi_{\rm I} \quad , \quad \delta_Q \Psi_{\rm I} = -\,\epsilon_{\rm I}\,(\,\partial_\tau \mathrm{X}\,) \;+\; \epsilon_{\rm I}\,F \quad , \\
&\delta_Q F = -\,i\,\epsilon_{\rm I}\,(\,\partial_\tau\Psi_{\rm I}\,) \quad .
\end{aligned} \tag{19}
$$

Finally there is the $N = 1$ theory

$$
\mathcal{S}_{ex}^{N=1} = \int d\tau \,\Big[\, \tfrac{1}{2}(\,\partial_\tau \mathrm{X}\,)(\,\partial_\tau \mathrm{X}\,) \;+\; i\tfrac{1}{2}\Psi\partial_\tau\Psi \,\Big] \quad , \tag{20}
$$

with transformation laws given by

$$
\begin{aligned}
\delta_Q \mathrm{X} &= i\,\epsilon\,\Psi \quad , \\
\delta_Q \Psi &= -\,\epsilon\,(\,\partial_\tau \mathrm{X}\,) \quad .
\end{aligned} \tag{21}
$$

It can be seen in all of these explicit cases, the equations of motion imply

$$
\partial_\tau\partial_\tau X \;=\; 0 \quad , \quad \partial_\tau\Psi \;=\; 0 \quad , \tag{22}
$$

where we have suppressed the O(N) index on the NRS spinor in order to discuss all the cases uniformly. For the $N = 2$,4 cases where there are required to be auxiliary fields present to close the algebra, their equations of motion imply that they should vanish on-shell. These results for the equations of motion in the special cases of $N =$ 2,4 suggest that in the general case of arbitrary N in (13) and (14) we should impose the conditions

$$
\mathcal{F}_l{}^j \;=\; 0 \quad , \quad \Lambda_{\hat{k}}{}^j \;=\; 0 \quad , \tag{23}
$$

to define the on-shell theories. In the case of arbitrary N, however, there are no solutions of (15) that possess equal numbers of fermions and bosons. Thus the requirement of equality of bosons and fermions needed for an off-shell supersymmetry is not satisfied in the general case of arbitrary N by a truncation. In the general case, it is not possible to write an appropriate action, either with or without the truncation.

The problem of finding an off-shell representation, such as in (14), which does not permit the writing of an appropriate action is well-known to superfield supergravity theories. For these it has been found that all the conformal degrees of freedom occur with a superfield called the "conformal prepotential." Using the conformal prepotential alone does not permit the writing of an action whose equations of motion correspond to those of the usual Einstein-Hilbert action. To do this requires additional supermultiplets called "compensators." It is thus natural to try a similar solution here.

In this case, we have suggested another route to obtaining an action. This begins with the introduction of a second supermultiplet that we have named the "Universal Spinning Particle Momentum Multiplet" with component fields ($\pi_{\rm I}$, $\mu_i{}^{\hat k}$, P, $\mathcal{G}_i{}^j$) that possess the supersymmetry variations given by

$$
\begin{aligned}
\delta_Q \, \pi_{\rm I} ~&=~ \epsilon_{\rm I} \, {\rm P} ~+~ {\rm d}^{-1} \epsilon_{\rm K} \, (f_{\rm K I})_j{}^i \, \mathcal{G}_i{}^j \quad , \\
\delta_Q \, \mu_i{}^{\hat k} ~&=~ -\, \epsilon_{\rm K} \, (L_{\rm K})_k{}^{\hat k} \, \mathcal{G}_i{}^k ~+~ {\rm d}^{-1} \epsilon_{\rm K} \, (L_{\rm I})_i{}^{\hat k} \, (f_{\rm I K})_k{}^l \, \mathcal{G}_l{}^k \quad , \\
\delta_Q \, {\rm P} ~&=~ -\, i \, 2 \epsilon_{\rm I} \, \partial_\tau \pi_{\rm I} \quad , \\
\delta_Q \, \mathcal{G}_i{}^j ~&=~ -\, i \, 2 \, \partial_\tau \, [\, \epsilon_{\rm J} \, (f_{\rm I J})_i{}^j \, \pi_{\rm I} ~+~ \epsilon_{\rm K} \, (R_{\rm K})_{\hat k}{}^j \, \mu_i{}^{\hat k} \,] \quad ,
\end{aligned}
\tag{24}
$$

and where the restrictions given by $\mathcal{G}_i{}^i = (R_{\rm I})_{\hat k}{}^i \mu_i{}^{\hat k} = 0$ must be satisfied.

The use of this second representation allows us to write an action that seems to make progress toward the problem of writing spinning particle theories with an arbitrary degree of extended supersymmetry.

$$
\begin{aligned}
\mathcal{L} = ~& -\, [\, i{\rm d}^{-1} \, \widetilde{\mu}_i{}^{\hat k} \partial_\tau \mu_i{}^{\hat k} ~+~ i \, \pi_I \partial_\tau \pi_I ~+~ \tfrac{1}{2} {\rm P}^2 ~+~ \tfrac{1}{2} {\rm d}^{-1} \, (\mathcal{G}_i{}^j \mathcal{G}_i{}^j) \,] \\
& +~ [~ -\, i \, \Psi_I \, (\, \partial_\tau \pi_I \,) ~+~ {\rm P}(\, \partial_\tau {\rm X} \,) ~+~ {\rm d}^{-1} \mathcal{G}_i{}^j \mathcal{F}_j{}^i ~+~ i{\rm d}^{-1} \mu_i{}^{\hat k} \Lambda_{\hat k}{}^i \,] \quad .
\end{aligned}
\tag{25}
$$

Variation of this Lagrangian with respect to all of the functions that appear in it leads to,

$$
\begin{aligned}
\delta\mu : &\qquad i \, 2 \partial_\tau \mu ~-~ i \, \Lambda ~&=~ 0 \quad , \\
\delta\pi_I : &\qquad i \, 2 \partial_\tau \pi ~+~ i \partial_\tau \Psi ~&=~ 0 \quad , \\
\delta{\rm P} : &\qquad {\rm P} ~-~ \partial_\tau {\rm X} ~&=~ 0 \quad , \\
\delta\mathcal{G} : &\qquad \mathcal{G} ~-~ \mathcal{F} ~&=~ 0 \quad , \\
\delta\Psi : &\qquad -\, i \partial_\tau \pi ~&=~ 0 \quad , \\
\delta\mathcal{F} : &\qquad \mathcal{G} ~&=~ 0 \quad , \\
\delta\Lambda : &\qquad \mu ~&=~ 0 \quad , \\
\delta{\rm X} : &\qquad \partial_\tau {\rm P} ~&=~ 0 \quad .
\end{aligned}
\tag{26}
$$

(Once again we have suppressed the indices on the fields for the sake of simplicity.)

Clearly in the bosonic sector we see

$$\begin{aligned} &\partial_\tau \mathrm{P} = \ 0 \ \& \ \mathrm{P} \ - \ \partial_\tau \mathrm{X} \ = \ 0 \ \rightarrow \ \partial_\tau \partial_\tau \mathrm{X} \ = \ 0 \quad , \\ &\mathcal{F} = \ \mathcal{G} \ = \ 0 \quad , \end{aligned} \tag{27}$$

and in the fermionic sector we find

$$\begin{aligned} i \, \partial_\tau (\pi \ + \ \tfrac{1}{2} \, \Psi) \ = \ 0 \quad , \quad - \, i \partial_\tau \pi \ = \ 0 \quad , \\ \mu \ = \ \Lambda \ = \ 0 \quad . \end{aligned} \tag{28}$$

If we combine the first two equations in (28) we arrive at the condition $i\partial_\tau \Psi = 0$. Comparing all of the results in (27) and (28) to those in (22) and (23), we see that the action in (25) succeeds in giving the correct equations of motion...with only one possible subtlety.

This subtlety involves the first equation of (28). This equations shows that the two functions Ψ and π can at most differ from each other by a zero mode. If this difference is negligible, then the action in (25) seems to be a suitable candidate to describe off-shell spinning particles. If this is not the case, then additional modifications are required. One way to attack these is to perform an analysis based on Dirac quantization. This is research that is presently underway.

4 THE N = 8 SPINNING PARTICLE/ SUPERGRAVITY SURPRISE

It is possible to consider the case of $N = 8$ within the context of the spinning particle models we described in the last section. For these, the appropriate γ-matrices have $d_N = 256$. Additionally, if we consider the spaces $\mathcal{U}$ and $\mathcal{M}$ defined in (15), they possess an interesting structure.

$$\begin{aligned} \{\mathcal{U}\} \ &= \ \{P_+ , \, P_+ \gamma^{\mathrm{I}_1 \mathrm{I}_2} P_+ \ , \, P_+ \gamma^{\mathrm{I}_1 \mathrm{I}_2 \mathrm{I}_3 \mathrm{I}_4} P_+ \} \quad , \\ \{\mathcal{M}\} \ &= \ \{P_+ \gamma^{\mathrm{I}} P_- \, , \ P_+ \gamma^{\mathrm{I}_1 \mathrm{I}_2 \mathrm{I}_3} P_- \} \quad , \end{aligned} \tag{29}$$

where the 4-form on the first line above is necessarily self-dual, i. e. $f_{\mathrm{I}_1\mathrm{I}_2\mathrm{I}_3\mathrm{I}_4} = \frac{1}{4!}\epsilon_{\mathrm{I}_1\mathrm{I}_2\mathrm{I}_3\mathrm{I}_4\mathrm{I}_5\mathrm{I}_6\mathrm{I}_7\mathrm{I}_8} f_{\mathrm{I}_5\mathrm{I}_6\mathrm{I}_7\mathrm{I}_8}$ and $f_{\mathrm{I}_1\mathrm{I}_2\mathrm{I}_3\mathrm{I}_4} \equiv P_+ \gamma_{\mathrm{I}_1\mathrm{I}_2\mathrm{I}_3\mathrm{I}_4} P_+$.

It is simple counting argument to note that the degeneracies of the elements in $\mathcal{U}$ go as 1, 28, and 35. The obtaining of the thirty-five is due to the duality condition.

Similarly, the degeneracies of the elements of $\mathcal{M}$ go as 8 and 56. We can go through a similar argument with regards to the elements of $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{M}}$. The only major difference is that the 4-form in $\widehat{\mathcal{U}}$ is anti-self-dual.

To the veteran supergravity researcher, the numbers 1, 28, and 35 along with 8 and 56 are striking because these are very reminiscent of the number of fields that occur in the 4D, $N = 8$ supergravity multiplet. However, these numbers also refer to fields of different spin. So at first it seems highly unlikely that their appearance in our present context is related to 4D, $N = 8$ supergravity.

In order to make this connection more closely we should also note that the numbers 1, 28, and 35 along with 8 and 56 in a sense correspond to half of 4D, $N = 8$ supergravity. Since these are associated with our one-dimensional construct, in order to get a complete 4D, $N = 8$ supergravity spectrum we ultimately expect the appearance of a perhaps a two dimensional construct similar to a string. Similarly, since 4D, $N = 8$ supergravity is the toric reduction of 11D, $N = 1$ supergravity, it is possible that the result of this section is the herald of some connection to M-theory. In the past, we have conjectured there may exist an NSR type of formulation of M-theory. The results in this section lend some support of this.

We have also spent some effort investigating realizations of 1D representations of supersymmetry from another viewpoint [4,5]. It is known that super Virasoro algebras make their appearance in superstring theory. We have also been undertaking a study of the super Virasoro algebras based on model-independent geometrical realizations. We have seen how a set of vector fields constructed over the superspace with coordinates $(\tau,\, \zeta^I)$ naturally possesses a representation of the centerless super Virasoro algebra. Furthermore, [5] by use of the co-adjoint representation, it has been seen that generalized geometrical realization implies that there is a relation between the order of a p-form that appears as a generator and the "spin" s of its co-adjoint field. This relation takes the form

$$s ~=~ \tfrac{1}{2}\,(\, 4 ~-~ p \,) ~~~ . \qquad (30)$$

Let us call the elements of $\mathcal{U}$ and $\widehat{\mathcal{U}}$ automorphic forms. This is an appropriate name for these since they may be thought of as linear maps acting to map the spaces of the definite chirality spinors of the γ-matrices back into themselves. Similarly, we can call the elements of $\mathcal{M}$ and $\widehat{\mathcal{M}}$ homomorphic forms because they act as linear maps taking spinors of one chirality and mapping them onto the space of the opposite chirality. All of the elements of $\mathcal{U}$, $\widehat{\mathcal{U}}$, $\mathcal{M}$ and $\widehat{\mathcal{M}}$ are indeed forms. So according to the relation in (30) we can assign a "spin" to each of them. This is done in the table below.

Algebraic Element	Spin	Degeneracy
$\mathcal{U}(\mathbf{I})$	2	1
$\widehat{\mathcal{U}}(\mathbf{I})$	2	1
$\mathcal{M}(f_{\rm I})$	3/2	8
$\widehat{\mathcal{M}}(\widehat{f}_{\rm I})$	3/2	8
$\mathcal{U}(f_{\rm IJ})$	1	28
$\widehat{\mathcal{U}}(\widehat{f}_{\rm IJ})$	1	28
$\mathcal{M}(f_{\rm I_1I_2I_3})$	1/2	56
$\widehat{\mathcal{M}}(\widehat{f}_{\rm I_1I_2I_3})$	1/2	56
$\mathcal{U}(f^{-}_{\rm I_1I_2I_3I_4})$	0	35
$\widehat{\mathcal{U}}(\widehat{f}^{+}_{\rm I_1I_2I_3I_4})$	0	35

Thus we seem to find the following interesting result. Each element in $\mathcal{U}$, $\widehat{\mathcal{U}}$, $\mathcal{M}$ and $\widehat{\mathcal{M}}$ can be associated with one of the fields that appears in 4D, $N = 8$ supergravity!

5 CONCLUSIONS

We believe that this simplest of supersymmetrical systems has important lessons for as yet unsolved problems in the topic. In closing we wish to reiterate what has been demonstrated. Firstly, for spinning particle actions, up to the issue of a zero mode, an off-shell formulation seems at hand. Obtaining the off-shell representation of the spinning particle fields was possible because these fields are in one-to-one correspondence with the elements of certain real Clifford algebras, the $\mathcal{GR}(N, d)$ algebras.

Due to this relation to Clifford algebras, we have not just found one off-shell representation of the spinning particle model. Instead we have found an infinite number of such representations. The key point is that once one has constructed the minimal off-shell representation of the $d_N \times d_N$ matrices in (4), there are an infinite number of increasingly larger sets of $d \times d$ matrices that also satisfy the relations in (5). These representation must be non-minimal representations of the spinning particles possessing larger and larger set of auxiliary fields.

We also find it intriguing that these real Clifford algebras make their appearance in this way. Although we do not have more direct evidence to support the following conjecture, we find this very suggestive that perhaps KO-theory, which is also based on real Clifford algebras, is playing some role in determining the representation theory of spinning particles. Additionally, from the discussion in the final chapter, there may be some way in which off-shell spinning particles, KO-theory together with the representation theory of super Virasoro algebras is connected with the theory of 4D, $N = 8$ supergravity. Due to this, we are hopeful about thepossibility of an NSR formulation of M-theory. Perhaps it is useful to recall the 1D nature of the M(atrix) model formulation of M-theory in this context. But this work is based on a non-linear realization of a Green-Schwarz approach by way of comparsion. Finally, we end with one more observation and conjecture.

Let us imagine that all supersymmetrical theories possess an off-shell representation. To be completely clear about this we mean an off-shell representation like that in (13). Such an off-shell representation may *not* by itself permit an appropriate action. This is exactly what we saw for the spinning particle. We believe this to be a very reasonable assumption. Why? As is well known in superspace approaches to supergravity, if one only places conventional constraints on the theory, by definition the component fields that arise are off-shell.

Now let us take such a theory and perform toroidal compactifications on all of the bosonic coordinates *except* the temporal one. Effectively this will lead to a 1D theory that must, however, maintain all of the supersymmetry apparent in the higher dimension. If there is a universality of the 1D representations that includes the spinning particle ones we have described, then it should be governed by the real Clifford algebras we have seen in the case of the off-shell spinning particles. In this case, the representation theory of the higher dimensional off-shell supergravity theories are contained in the 1D theories we have constructed in this work and it is possible that there is some type of encoding of all off-shell supersymmetrical theories contained in 1D supersymmetrical theories.

On this basis, we have looked at the issue of the possibility of an off-shell formulation of 4D, $N = 8$ supergravity (or alternately the 11D, $N = 1$ supergravity limit of low energy M-theory) and concluded that the smallest possible representation contains 32,768 bosons + 32,768 fermions. We conjecture that there exists an off-shell 11D, $N = 1$ supergravity theory possessing these numbers of bosonic and fermionic degrees of freedom. Should this off-shell 4D, $N = 8$ (or alternately 11D, $N = 1$) supergravity multiplet exist, it should prove to be the analog of the spinning particle multiplet in (13). Proving this will answer part of the challenge described by our title.

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